Effect of an Impurity on Grey Soliton Dynamics in Cigar-Shaped Bose-Einstein Condensate

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Abstract. - In a cigar shaped Bose-Einstein condensate, explicit solutions of the coupled mean-field equations, describing defect-grey soliton dynamics are obtained, demonstrating the coexistence of grey soliton and a localized defect. Unlike the case of dark soliton, where the defect trapping center has vanishing superfluid density, the moving grey soliton necessarily possesses a finite superfluid component at the defect location. The wave vector of the impurity is controlled by the velocity of the grey soliton, which has an upper bound. It is found that the presence of the impurity lowers the speed of the grey soliton, as compared to the defect free case, where it can reach the sound velocity. The grey soliton's energy gets substantially modified through its interaction with the defect, opening up the possibility of its control through defect dynamics.

Quite some time back, Gross formulated the mean field equations describing the macroscopic dynamics of defects and superfluid matter [1,2]. Akin to the Gross-Pitaevskii (GP) equation for Bose-Einstein condensate (BEC), these mean-field equations are well suited for describing the defect-BEC dynamics in a trap. On the theoretical side, a number of works explored the phenomenological features of BEC-impurity complex [3–7]. The dynamics of the BEC, trapped in an optical lattice, in presence of a localized impurity has been investigated [8]. The interactions of defect atoms in an optical lattice with a uniform BEC have also been studied for the understanding the dephasing effect on this system [9]. Recently, Roberts and Rica [10], investigated the behavior of the impurity field in a BEC and identified the parameter domains for the formation of a crystal of impurity fields and supersolid phases. The fact that the geometry of this complex can be effectively manipulated, has motivated the study of the BEC-defect dynamics in lower dimensions. Cigar shaped BEC has received special attention, because of the possibility of identifying exact solutions. It was found that, even approximate solutions in one dimension, well represent this coupled system [11].

Experimental realization of dark and bright soliton [12–17], and soliton trains [18] has given impetus to the study of the interaction between defect-soliton system. In 1997, Konotop *et al.* [19], studied the dark soliton-impurity complex by means of a modified adiabatic approximation and observed differences between dark and bright soliton dynamics perturbed by a point defect. In Ref. [20], Frantzeskakis *et al.*, considered a static impurity and investigated the interaction of dark soliton with localized impurities. It was found that the dark soliton can get reflected or transmitted by a repulsive impurity. Self-trapping of impurities was studied, for the case of both repulsive and attractive interactions [11]. The soliton-defect dynamics was also investigated by Goodman *et al.* [21]. In a recent work, it has been shown that a dissipation source can be used to generate dark, bright, gap and ring dark solitons, by controlling their phase and amplitude [22]. Very recently, Dries *et al.* [23], investigated the effects of impurities on collective dipole motion of the BEC and characterized the breakdown of superfluidity of the trapped cloud in both 3D Thomas-Fermi and quasi-1D weakly interacting regimes. Apart from dark and bright solitons, GP equation supports a complex envelope soliton, analogous to the Bloch soliton in magnetic systems [24,25]. This grey soliton has recently been observed through a controlled density engineering method [26,27]. Its velocity can take values from zero to the sound velocity

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and has a dispersion very different from that of dark soliton [28,29]. It is then natural to inquire the nature of the interaction between grey soliton and a point like defect.

In a recent experiment, it was observed that, when the velocity of the impurity atom was made to decrease below the condensate sound velocity, the collisional cross section decreased abruptly [30]. This indicates the better applicability of mean-field dynamics in this regime. Keeping this in mind, we study here the BEC-defect complex in the mean-field approximation, in a quasi-one dimensional scenario, where the defect atom moves in the condensate with constant velocity. As will be seen below, for this system both defect and soliton velocities are bounded from above. The general case is investigated, where defect and the condensate atoms have different masses. The density of the impurity atom is assumed to be small, so that the effect of the impurity on BEC excitation spectrum remains negligible. We find exact solutions of the coupled mean-field equations, describing the interaction of a grey soliton with a localized defect. The local minimum of the soliton in the defect location has a non-vanishing superfluid density. The presence of the defect lowers the grey soliton's velocity, as compared to the defect free case, where the maximum velocity is the sound velocity. Physically, the presence of the defect atom acts like a drag on the grey soliton, which results in a decrease of its speed from the sound velocity. The stability of the above solution is investigated using the well known Vakhitov-Kolokolov (VK) criteria [31,32] and found that the obtained solutions are stable. It is also found that the defect affects the energy of the grey soliton considerably, opening up the possibility of controlling the grey soliton through defect dynamics. We then compared these results with pure dark and bright soliton-defect complex, obtained from the mean-field equations. The mean-field solutions require the masses of the impurity and that of the condensate atoms to be same, implying that the impurity is the same atom in a different hyperfine level. Unlike the previous case, no restriction is found on the velocity of the soliton or that of the defect.

In the model of Gross, the interaction between impurity and the condensate is treated in the Hartree approximation. The defect-condensate dynamics is described by the following coupled equations, where the wavefunctions of the condensate atoms and the defect are ψ_a and ψ_b, respectively:

$$\begin{aligned} i\hbar\frac{\partial\Psi_a}{\partial t} &= -\frac{\hbar^2}{2m_a}\vec{\nabla}^2\Psi_a + V_{ext}\Psi_a + g|\Psi_a|^2\Psi_a \\ &\quad +\kappa|\Psi_b|^2\Psi_a - \mu\Psi_a, \quad (1) \\ \text{and}\quad i\hbar\frac{\partial\Psi_b}{\partial t} &= -\frac{\hbar^2}{2m_b}\vec{\nabla}^2\Psi_b + V_{ext}\Psi_b + \kappa|\Psi_a|^2\Psi_b (2) \end{aligned}$$

Here, $g = \frac{4\pi\hbar^2 a}{m_a}$ is the strength of the atom-atom interaction and the strength of the interaction between atoms in the condensate and the impurity is given by, $\kappa = \frac{2\pi\hbar^2 a_{12}}{m_r}$, where, $m_r = \frac{m_a m_b}{m_a+m_b}$ is the reduced mass. The mass of the condensate atoms and that of the defect are m_a and m_b, respectively, with μ being the chemical potential. The condensate wavefunction is normalized to the number of atoms N and the defect wavefunction to unity.

For cigar shaped BEC, the three dimensional coupled GP equation can be transformed into quasi-one dimension: $\Psi_i(r,t) = \psi_i(x,t)f_i(y,z)$, $(i = a,b)$. The condensate atoms and the impurity can be made to experience the same trapping potential $V_{ext} = \frac{1}{2}m_a\omega_a^2(y^2+z^2)$, by tuning the axial frequencies of the atoms ω_a and that of the impurity ω_b, which differs by a factor of $\sqrt{m_a/m_b}$ [33]. As is known, two different masses can bring in transverse separation due to gravity [33,34]. We assume that, this separation is negligible. In the reduction to quasi-one dimension, taking into account the tight harmonic trap, one needs two different widths for the transverse Gaussian profiles [35–37]:

$$\begin{aligned} f_a(y,z) &= \frac{1}{\sqrt{\pi}a_\perp}e^{-(y^2+z^2)/2a_\perp^2}, \quad (3) \\ \text{and}\quad f_b(y,z) &= \frac{1}{\sqrt{\pi}b_\perp}e^{-(y^2+z^2)/2b_\perp^2}, \quad (4) \end{aligned}$$

where $a_\perp = \sqrt{\frac{\hbar}{m_a\omega_a}}$ and $b_\perp = \sqrt{\frac{\hbar}{m_b\omega_b}}$. The three dimensional coupled equations can be mapped into one dimension by minimizing the action functional after integrating over the transverse degrees of freedom. In the weak coupling scenario, the coupled equations in one dimension are given by [38,39],

$$i\hbar\frac{\partial\psi_a}{\partial t} = -\frac{\hbar^2}{2m_a}\frac{\partial^2\psi_a}{\partial x^2} + \tilde{g}|\psi_a|^2\psi_a + \tilde{\kappa}|\psi_b|^2\psi_a - \mu\psi_a, \quad (5)$$

$$i\hbar\frac{\partial\psi_b}{\partial t} = -\frac{\hbar^2}{2m_b}\frac{\partial^2\psi_b}{\partial x^2} + \tilde{\kappa}|\psi_a|^2\psi_b, \quad (6)$$

where $\tilde{g} = \frac{1}{2\pi a_\perp^2}g$ and $\tilde{\kappa} = \frac{1}{\pi(a_\perp^2+b_\perp^2)}\kappa$. We now drop the tilde for notational convenience and study the general case, where the mass of the condensate atoms and that of the impurity are different. The following envelope profiles lead to the description of grey soliton impurity complex: $\psi_a = \sqrt{\sigma_a}e^{i\chi}$ and $\psi_b = \sqrt{\sigma_b}e^{ikx-i\omega t}$. It is worth emphasizing that for the grey soliton to exist, the defect should necessarily possess the plain wave component e^{ikx}. Current conservation and consistency conditions yield:

$$\frac{\hbar}{m_a}\chi' = u(1-\frac{\sigma_0}{\sigma_a}) \quad\text{and}\quad \hbar k = m_b u. \quad (7)$$

The soliton satisfies the boundary condition that, asymptotically as $\sigma \to \sigma_0$, the phase variation vanishes. Here, $\sigma_0 = \frac{\mu}{g}$, is the equilibrium density of the condensate. u is the velocity of the soliton. The real part of Eq. (5) can then be cast in a convenient form, in terms of the densities:

$$\sigma_a\sigma_a'' - \frac{1}{2}\sigma_a'^2 + \left(\frac{1}{2}m_a u^2 + \mu\right)\sigma_a^2 \quad - \quad g\sigma_a^3 - \kappa\sigma_b\sigma_a^2$$

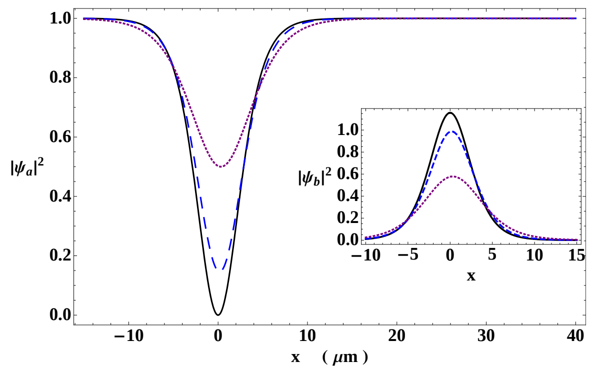

Fig. 1: The figure shows the variation of the grey soliton profile for different values of θ. The solid (black), dashed (blue) and dotted (purple) lines correspond to $\theta = 0$, $\pi/8$ and $\pi/4$, respectively. Inset depicts the corresponding density profiles of the impurity atom. The densities ($|\psi_i|^2, i = a, b$) are measured in the unit of σ_0. The parameter values are: $\sigma_0 a = 0.1$, $\omega_a = 2\pi \times 120$ Hz, $\omega_b = 2\pi \times 122$ Hz, $m_a = 1.44312 \times 10^{-25}$ Kg, $m_b = 1.40995 \times 10^{-25}$ Kg, $a = 213a_0$, $a_{12} = 99a_0$ ($a_0 = 5.3 \times 10^{-11}$ m, is the Bohr radius).

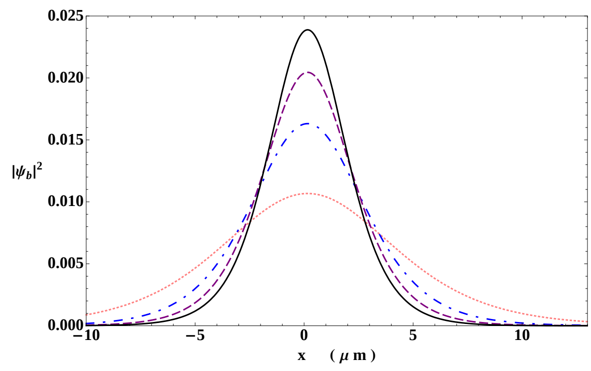

Fig. 2: The figure shows the variation of the density of the impurity ($|\psi_b|^2 = \sigma_b$) for different values of atom-impurity coupling κ. The ratio m_b/m_a is kept fixed at 0.997 for all the values of g/κ. The dotted (pink), dash-dotted (blue), dashed (purple) and solid (black) lines correspond to $a_{12} = 50a_0, 100a_0, 150a_0$ and $200a_0$, respectively. One can see that as a_{12}, i.e., κ increases, the density of the impurity increases. The parameter values used for this figure are same as those of the Fig. (1).

$$+ \quad \frac{1}{2} m_a u^2 \sigma_0^2 = 0. \quad (8)$$

The following densities exactly solve Eq. (8),

$$\sigma_a = \sigma_0 - \sigma_0 \cos^2\theta \mathrm{sech}^2\left[\frac{\cos\theta}{\zeta}(x - ut)\right], \quad (9)$$

$$\text{and } \sigma_b = b^2 \mathrm{sech}^2\left[\frac{\cos\theta}{\zeta}(x - ut)\right], \quad (10)$$

where, $b^2 = \frac{\cos\theta}{2\zeta}$ with $\sigma_0 = \frac{1}{2\zeta\cos\theta(g/\kappa - m_b/m_a)}$ and $\hbar\omega = \frac{\hbar^2 k^2}{2m_b} - \frac{\hbar^2}{2m_b\zeta^2}\cos^2\theta + \kappa\sigma_0$. The impurity atom rests on the minimum of the grey soliton. Notice that for σ_0 to be a positive definite, $\left(\frac{g}{\kappa} - \frac{m_b}{m_a}\right) > 0$. Repulsive atom-atom interaction leads to $\kappa < \frac{g m_a}{m_b}$, provided the atom-impurity interaction is also repulsive. If either of these two interactions is attractive, the localized solutions cease to exist. The presence of localized solitons crucially depends on the balancing between nonlinearity and dispersion effects. The fact that the condensate has the self-interaction, apart from its interaction with the impurity, whereas, the impurity is devoid of self-interaction, leads to the dominant role of the condensate profile over the defect. It can be shown that for a significant increase in the value of g/κ, corresponding to very weak impurity coupling, the density of the impurity is very small and has a marginal effect on the grey soliton. From Eq. (10), the following two cases (a) $u = 0$, i.e., static solution and (b) $u \neq 0$, needs to be treated separately. When $u = 0$, the limit $\kappa \to 0$ leads to vanishing of the impurity wave-function and the grey soliton tends towards a dark soliton $\psi_a = \sqrt{\sigma_0}\tanh(\frac{x}{\zeta})$. However, this limiting case leads to a divergence problem in the healing length $\zeta = \frac{\hbar}{\sqrt{m_b\kappa\sigma_0}}$, and hence, is not admissible. For the second case, it is clear from Eq. (10), that for $u \neq 0$, the limit $\kappa \to 0$ is unphysical, since the density of the impurity becomes imaginary. In this case, the solutions exist only when $\kappa \geq \frac{u^2 m_a^2}{m_b\sigma_0}$. It is worth mentioning that it has been recently found that, if the mass of the impurity atom is too heavy and strongly interacting, the impurity can break the condensate into two parts, where the mean-field theory breaks down [40].

Interestingly, the sound velocity in this case is found to be: $c_w = \zeta\sigma_0 g/\hbar$. The velocity angle, also known as the Mach angle, is given by,

$$\theta = \sin^{-1}\frac{u}{u_s}, \quad (11)$$

where the maximum velocity of the grey soliton is $u_s = c_w \frac{m_b\kappa}{m_a g}$ for $\theta = \pi/2$. In the present case, it is less than the sound velocity of the condensate. In the limit $\kappa \to g m_a/m_b$, though the soliton maximum velocity $u_s \to c_w$, the equilibrium density of the condensate σ_0 diverges. Therefore, the limit $\kappa \to g m_a/m_b$ is unphysical. The obtained solutions exist within the limit $\frac{u^2 m_a^2}{m_b\sigma_0} \leq \kappa < g m_a/m_b$. Superfluid density vanishes at $\theta = 0$, which corresponds to a static soliton. For a moving soliton, the superfluid density at the defect location is finite.

In the presence of impurity, both the healing length and the maximum soliton velocity get modified. The existence of the localized soliton depends on the balance between nonlinearity and dispersion. The atom-impurity nonlinear interaction contributes to the balancing effect. As can be seen explicitly, the maximum velocity depends on the density of the condensate, interaction between the atoms in the condensate, as well as the interaction between the atoms and that of the impurity. The presence of impurity atom lowers the grey soliton's velocity. Interestingly,

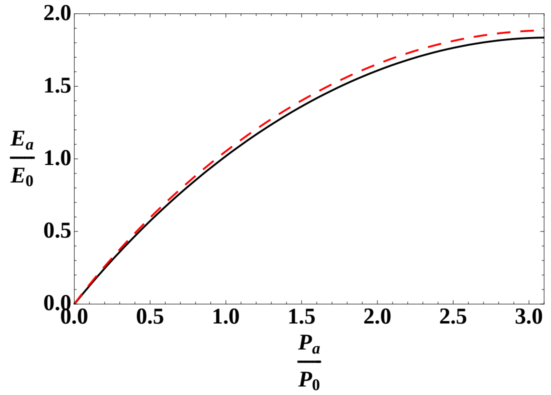

Fig. 3: Dispersion relation of the grey soliton (solid line), in presence of an impurity, with the dashed line above, depicting the same for pure grey soliton [29]. The energy and momentum are measured in the units of E_0 and P_0, respectively. The parameter values, used for this figure are same as those of the Fig. (1).

the effect of the impurity leads to the appearance of the dimensionless ratio $\frac{m_b}{m_a}$, in the maximum velocity expression.

The wavefunctions ψ_a and ψ_b can also be deduced:

$$\psi_a(x,t) = i\sqrt{\sigma_0}\sin\theta + \sqrt{\sigma_0}\cos\theta\tanh\left(\frac{\cos\theta}{\zeta}(x-ut)\right), \tag{12}$$

$$\psi_b(x,t) = ib\,\mathrm{sech}\left[(\frac{\cos\theta}{\zeta}(x-ut)\right) e^{ikx-i\omega t}. \tag{13}$$

It is worth pointing out that the coupled complex envelope and bright soliton solutions obtained here are of similar type, found earlier in the context of two component BEC [41] and in boson-fermion mixtures [42]. Fig. (1) depicts the grey soliton profiles, for various values of the Mach angle. Corresponding defect profiles are shown in the inset. It is worth observing that the localized defect resides at the minimum of the grey soliton, similar to the experimental observation in a two component BEC, when one component has less number of atoms [34]. Notice that, as θ increases, the amplitudes of the grey soliton, as well as of the defect decrease.

Fig. (2) shows the variation of the impurity density for different values of a_{12}. The dotted (pink), dash-dotted (blue), dashed (purple) and solid (black) lines correspond to $a_{12} = 50a_0, 100a_0, 150a_0$ and $200a_0$, respectively. One can see that as a_{12}, i.e., κ increases, the density of the impurity increases. For small values of κ, the impurity density is very small, which implies that the effect of the impurity on grey soliton is marginal.

The stability of the grey soliton is investigated using the well known criteria of Vakhitov and Kolokolov [31,32] for non-linear Schrödinger type equation. It is known from this criterion that the solution is stable and unstable if $\frac{\partial N}{\partial \mu} > 0$ and $\frac{\partial N}{\partial \mu} < 0$, respectively. When $\frac{\partial N}{\partial \mu} = 0$, the solutions are found to be marginally stable. We obtained the exact expression of the number of atoms from the normalization condition of the condensate wavefunction,

$$N = \int_{-\infty}^{\infty} |\psi_a|^2 dx = \frac{2\hbar}{m_b\kappa}\sqrt{\frac{m_b\kappa\mu - u^2 m_a^2 g}{g}}. \tag{14}$$

In order to obtain the stability condition, we calculate,

$$\frac{\partial N}{\partial \mu} = \frac{2\hbar}{g}\left(\frac{m_b\kappa\mu - u^2 m_a^2 g}{g}\right)^{-\frac{1}{2}}. \tag{15}$$

The restriction on κ gives $\frac{\partial N}{\partial \mu} > 0$. Therefore, the obtained solution is found to be stable within the allowed range of κ.

The energy and momentum of the solitary wave and defect can be obtained by removing the background contribution, which ensures the convergence of the integrals,

$$E_a = \frac{\hbar^2}{2m_a}\int \frac{\partial\psi_a^*}{\partial x}\frac{\partial\psi_a}{\partial x}dx + \frac{1}{2}g\int(\sigma_a-\sigma_0)^2 dx + \kappa\int(\sigma_a-\sigma_0)\sigma_b dx. \tag{16}$$

Using the density profiles, obtained in Eq. (12) and (13), Eq. (16) yields,

$$E_a = \frac{2\hbar^2\sigma_0}{3m_a\zeta}\cos^3\theta + \frac{2}{3}g\sigma_0^2\zeta\cos^3\theta - \frac{2}{3}\kappa\sigma_0\cos^2\theta. \tag{17}$$

The three terms in the energy expression respectively represent, the kinetic, self-interaction and the interaction energy between the soliton and defect. As is evident, the energy of the grey soliton gets substantially modified because of the presence of the impurity implying the possibility of controlling soliton dynamics through the defect. The momentum of the grey soliton remains unchanged: $P_a = \sigma_0\hbar\left(\pi\frac{u}{|u|} - 2\theta - \sin 2\theta\right)$. The dispersion relation is shown in Fig. (3), which also shows the same for that of the pure grey soliton for comparison. The solid line depicts the dispersion of the grey soliton in presence of an impurity, whereas, the dotted line is for pure grey soliton. Both the energy and momentum are measured in the unit of $E_0 = \hbar\omega_a\sqrt{\sigma_0 a}\sigma_0 a_\perp$ and $P_0 = \sigma_0\hbar$ [29]. It is observed that for long wavelength excitations, the Lieb mode associated with the BEC-defect complex does not differ significantly with that of the Lieb mode in defect free case at low momenta, and hence with the Bogoliubov mode. Hence, employing a single defect, one can control the velocity of the grey soliton, which is found to be less than that of the pure grey soliton.

For the sake of completeness, we have computed the energy and momentum of the impurity. The energy of the impurity with respect to the background σ_0

$$E_b = \frac{\hbar^2 k^2}{2m_b} - \frac{\hbar\cos^2\theta}{6m_b\zeta^2} - \frac{2}{3}\kappa\sigma_0\cos^2\theta, \tag{18}$$

with the corresponding momentum: $P_b = \hbar k$.

The solutions of the coupled mean-field equations also admit dark and bright solitons, when both masses are equal ($m_a = m_b = m$):

$$\psi_a(x,t) = \sqrt{\sigma_0}\tanh[\frac{1}{\zeta}(x-ut)]e^{ikx-i\omega t}, \tag{19}$$

$$\text{and} \quad \psi_b(x,t) = b\,\mathrm{sech}[\frac{1}{\zeta}(x-ut)]e^{ikx-i\omega t}, \tag{20}$$

with $mu = \hbar k$ and the healing length $\zeta = \frac{\hbar}{\sqrt{m\kappa\sigma_0}}$. One notes that unlike the previous case, a kinematic phase is associated with both atoms and the impurity. The amplitude of the impurity is given by, $b = \frac{1}{\sqrt{2\zeta}}$ with $\sigma_0 = \frac{1}{2\zeta(\frac{g}{\kappa}-1)}$. As is evident, for the solutions to exist, the atom-atom coupling must be greater than atom-impurity coupling. Furthermore, the solutions exist only when both the interactions are repulsive or attractive. As compared to the previous case, the velocity of the dark soliton u can take any finite value.

In summary, we have obtained exact solutions for the grey soliton impurity complex in a cigar-shaped geometry. The impurity resides in the local minimum of the grey soliton, where the superfluid density is finite. The velocity of the grey soliton gets restricted, which in turn, controls the high momentum component associated with the impurity. We observed that the presence of the impurity leads to a reduction of grey soliton velocity, which for the defect free condensate can reach sound velocity. It is found that the impurity modifies the energy of the solitary wave, opening up the possibility to control the grey soliton dynamics through impurity. The obtained solutions are found to be stable, within the allowed range of κ, as per the VK criteria. Pure dark or bright soliton, having a vanishing superfluid component at the defect location, are identified, when defect and condensate atoms have identical masses. Unlike the previous case, there is no restriction on the velocity of the grey soliton. We hope that grey soliton defect complex and its dynamics can be observed with present laboratory setup. The response of this complex to trap and scattering length variations [43–45] is worth investigating, as well as the study of this complex in an optical lattice [46, 47].

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